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Dualizability and graph algebras

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Abstract

We characterize the finite graph algebras which are dualizable. Indeed, a finite graph algebra is dualizable if and only if each connected component of the underlying graph is either complete or bipartite complete (or a single point).

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1. Introduction

This paper serves two purposes: it provides a characterization of finite graph algebras which are dualizable, and it elaborates some techniques which promise to be useful in establishing that various finite algebras are not dualizable. Those techniques are also applied herein to sharpen some existing nondualizability results. It turns out that a finite graph algebra is dualizable if and only if each connected component of the underlying graph is either complete or complete bipartite (or a single point). This in turn is known to be equivalent to the graph algebra having a finitely axiomatizable equational theory.

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Graph algebras were introduced by C. Shallon in her dissertation [26] as a general framework for constructing finite algebras with unusual properties. Given a graph algebra, the underlying graph can easily be recovered.

The familiar dualities of Stone [27] for Boolean algebras, of Priestley [24] for bounded distributive lattices, and of Pontryagin [22] for Abelian groups exemplify the meaning of dualizable. These dualities fall under the broad umbrella of the theory of natural dualities. The survey of Davey [6] and the monograph of Clark and Davey [4] provide useful accounts of the theory of natural dualities. While the necessary definitions will be included below, we rely on these two sources for the underlying theorems and proofs as well as for the context of our results.

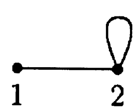
By an *algebra* we understand a nonempty set endowed with a system of finitary operations. We denote algebras in boldface, for example, $\boldsymbol{A}$, while the underlying set is denoted without emphasis, for example, A. We use, without explanation, common notions from the general theory of algebras such as homomorphism, subalgebra, direct product, and congruence relation. The monograph of McKenzie et al. [16] gives an account of the general theory of algebras; for the most part we follow the notation of that monograph.

Given a graph G, possibly with loops at some of its vertices, the algebra $\boldsymbol{A}(G)$, called the *graph algebra of* G, is the algebra with universe $V \cup \{0\}$, where V is the set of vertices of G and we insist that $0 \notin V$. The algebra $\boldsymbol{A}(G)$ has just one basic operation, which is binary and defined as follows:

$$u \cdot v = \begin{cases} u & \text{if } u, v \in V \text{ and an edge of } G \text{ joins } u \text{ and } v, \\ 0 & \text{otherwise.} \end{cases}$$

An example of a graph M and the multiplication table for the algebra $\boldsymbol{A}(M)$ are displayed below.

M



$\mathbf{A}(M)$

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

In this paper our concern is with dualities for certain classes generated by finite algebras. So our definitions will be specialized to this case.

Let $\boldsymbol{B}$ be a finite algebra. By an *alter ego* of $\boldsymbol{B}$ we mean a structured topological space $\mathbb{B}$ where the topology is the discrete topology on B and the additional structure consists of a system of (possibly infinitely many) finitary operations, finitary partial operations, and finitary relations on B each of which must be a subuniverse of the appropriate finite direct power of the algebra $\boldsymbol{B}$. Suppose now that $\boldsymbol{A}$ is (isomorphic to) a subalgebra of a direct power of $\boldsymbol{B}$ (which we denote $\boldsymbol{A} \in \boldsymbol{SP}\,\boldsymbol{B}$). Then $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ will be a topologically closed subuniverse of $\mathbb{B}^A$. We let $\mathbb{D}(\boldsymbol{A})$ denote the corresponding structured topological space and refer to it as the *dual* of $\boldsymbol{A}$ (with respect to $\mathbb{B}$). Likewise, given a structured topological space $\mathbb{X}$ which is (isomorphic to) a topologically closed substructure of some nontrivial power of $\mathbb{B}$ the set $\mathrm{Hom}(\mathbb{X},\mathbb{B})$ of continuous structure preserving maps from $\mathbb{X}$ into $\mathbb{B}$ is a nonempty subuniverse of the algebra $\boldsymbol{B}^X$. We denote the corresponding subalgebra by $\boldsymbol{E}(\mathbb{X})$ and refer to it as the *dual* of $\mathbb{X}$. Under the stipulations set out above, there is a natural embedding e of the algebra $\boldsymbol{A}$ into its double dual $\boldsymbol{E}(\mathbb{D}(\boldsymbol{A}))$. Indeed, e merely assigns to each $a \in A$ the evaluation map e_a defined via

$$e_a(\alpha) = \alpha(a) \quad \text{for all } \alpha \in \mathbb{D}(\boldsymbol{A}).$$

Generally, $\boldsymbol{E}(\mathbb{D}(\boldsymbol{A}))$ will have members that are not such evaluation maps and e will fail to map onto $\boldsymbol{E}(\mathbb{D}(\boldsymbol{A}))$. In the event that e maps onto $\boldsymbol{E}(\mathbb{D}(\boldsymbol{A}))$ we say that $\mathbb{B}$ *yields a duality on* $\boldsymbol{A}$. Finally, we call the algebra $\boldsymbol{B}$ *dualizable* provided it has an alter ego $\mathbb{B}$ so that $\mathbb{B}$ yields a duality on $\boldsymbol{A}$ for every algebra $\boldsymbol{A} \in \boldsymbol{SP}\ \boldsymbol{B}$.

If $\boldsymbol{B}$ is dualizable at all, then it is clear from the above construction that it is dualizable using the richest possible alter ego, the one equipped with all the appropriate operations, partial operations, and relations. Such an alter ego is called the *brute force* alter ego of $\boldsymbol{B}$. Thus, $\boldsymbol{B}$ fails to be dualizable if and only if there is $\boldsymbol{A} \in \boldsymbol{SP}\ \boldsymbol{B}$ such that the brute force alter ego of $\boldsymbol{B}$ does not yield a duality on $\boldsymbol{A}$.

We now discuss a generalization of the notion of dualizability which emerged in [5]. Let $\boldsymbol{B}$ be a finite algebra and κ be a cardinal. A *κ-alter ego* of $\boldsymbol{B}$ is a structured topological space $\mathbb{B}$ where the topology is the discrete topology on B and the additional structure consists of a system of (possibly infinitely many) operations, partial operations, and relations on B each of which must be a subuniverse of some direct power $\boldsymbol{B}^{\lambda}$ of the algebra $\boldsymbol{B}$, where λ is a cardinal smaller than κ. (Such operations, partial operations and relations are called *κ-algebraic* for $\boldsymbol{B}$.) Note that an ordinary alter-ego is an ω-alter ego.

Suppose now that $\boldsymbol{A} \in \boldsymbol{SP}\ \boldsymbol{B}$. Then, again, $\mathrm{Hom}(\boldsymbol{A},\boldsymbol{B})$ will be a topologically closed subuniverse of $\mathbb{B}^A$, and $\mathbb{D}(\boldsymbol{A})$ denotes the corresponding structured topological space. We refer to it as the *dual* of $\boldsymbol{A}$ (with respect to $\mathbb{B}$).

Likewise, suppose that $\mathbb{X}$ is a structured topological space which is isomorphic to a topologically closed substructure of some nontrivial power of $\mathbb{B}$. Then here too the set $\mathrm{Hom}(\mathbb{X},\mathbb{B})$ of continuous structure preserving maps from $\mathbb{X}$ into $\mathbb{B}$ is a nonempty subuniverse of the algebra $\boldsymbol{B}^X$. Here too we denote the corresponding subalgebra by $\boldsymbol{E}(\mathbb{X})$ and refer to it as the *dual* of $\mathbb{X}$. Again the mapping e which assigns each a to the evaluation map at a is an embedding of the algebra $\boldsymbol{A}$ into its double dual $\boldsymbol{E}(\mathbb{D}(\boldsymbol{A}))$. Extending the traditional terminology further, if this mapping e is onto, then we say that $\mathbb{B}$ *yields a κ-duality on* $\boldsymbol{A}$. The algebra $\boldsymbol{B}$ is *κ-dualizable* provided it has a κ-alter ego $\mathbb{B}$ so that $\mathbb{B}$ yields a κ-duality on $\boldsymbol{A}$ for every algebra $\boldsymbol{A} \in \boldsymbol{SP}\ \boldsymbol{B}$.

$\mathcal{Q}$ is a *finitely generated quasivariety* if $\mathcal{Q} = \boldsymbol{SP}\ \boldsymbol{B}$ for some finite algebra $\boldsymbol{B}$, that is, just in case each algebra in $\mathcal{Q}$ is (isomorphic to) a subalgebra of a direct power of $\boldsymbol{B}$, and $\mathcal{Q}$ is the class of all such algebras. (In general, a quasivariety is a class defined by quasiequations. An example of a quasiequation is $\forall x, y\ [x(yy) = x \Rightarrow xy = x]$, and this happens to hold in all graph algebras. However, we will not need this general definition.) A finitely generated quasivariety $\mathcal{Q}$ is *κ-dualizable* just in case $\mathcal{Q} = \mathbf{SP}\ \mathbf{B}$ for a finite, *κ-dualizable* algebra $\mathbf{B}$.

We introduce in this paper a strong form of nondualizability. A finitely generated quasivariety $\mathcal{Q}$ is *inherently non-κ-dualizable* provided that $\mathcal{R}$ is not κ-dualizable whenever $\mathcal{R}$ is a finitely generated quasivariety which contains $\mathcal{Q}$. A finite algebra $\mathbf{H}$ is *inherently non-κ-dualizable* if the quasivariety $\mathbf{SP}\ \mathbf{H}$ is inherently non-κ-dualizable. (We say an algebra is *inherently nondualizable* if it is inherently non-ω-dualizable.)

Thus a finite algebra $\mathbf{H}$ is inherently non-κ-dualizable just in case any finitely generated quasivariety $\mathcal{Q}$ is not κ-dualizable whenever $\mathbf{H} \in \mathcal{Q}$, or equivalently, in case $\mathbf{B}$ is non-κ-dualizable whenever $\mathbf{B}$ is a finite algebra with $\mathbf{H} \in \mathbf{SP}\ \mathbf{B}$. Once the inherent non-$\kappa$-dualizability of an algebra $\mathbf{H}$ has been established, not only do we know that $\mathbf{H}$ is not κ-dualizable, but we are in a position to show that a host of other finite algebras also fail to be κ-dualizable.

In Section 2 we develop some techniques, based on the ghost element method, for proving that certain finite algebras are inherently non-κ-dualizable for any κ. Even though it follows from work of Murskiĭ [20] and Davey and Werner [9] that almost all finite algebras are dualizable, we are able to prove that there is an absolute constant c such that the number of inherently non-κ-dualizable n-element groupoids is at least cn^{n^2-2} for all $n > 1$ and all cardinals κ. The ghost element method also allows us to prove that there are continuum many inherently non-κ-dualizable clones on $\{0, 1, 2\}$, each generating a 3-permutable, congruence distributive variety, extending results of Heindorf [11] and Idziak [12].

An algebra $\mathbf{A}$ is *finitely based* provided there is a finite set Σ of equations, each true in $\mathbf{A}$, such that every equation true in $\mathbf{A}$ is a logical consequence of Σ. A finite algebra with only finitely many basic operations may fail to be finitely based, see Lyndon [14], and the question of which finite algebras are finitely based has proved to be subtle. In the event that $\mathbf{A}$ is dualizable, we know that every algebra in $\mathbf{SPA}$ has a topologically compact dual. Thus dualizability may be viewed as a finiteness property of the quasivariety $\mathbf{SPA}$. This led to the following problem.

Problem. Is every finite dualizable algebra finitely based?

In 1976, Park [21] conjectured that every finite algebra with finitely many basic operations which belongs to a variety with a finite residual bound must be finitely based. This conjecture is still open, although it has been settled in the affirmative in case the algebra belongs to a congruence distributive variety [1], a congruence modular variety [15], or a congruence meet-semidistributive variety [30]. The following, perhaps more tractable, variant of the problem above is also open.

Problem. Is every finite dualizable algebra with only finitely many basic operations which belongs to a variety with a finite residual bound finitely based?

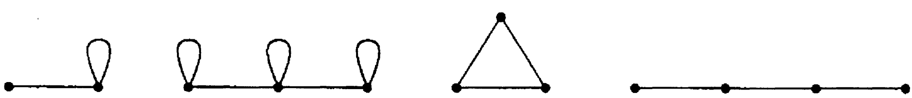
Shallon observed that the three-element nonfinitely based groupoid invented by Murskiĭ [19] is a graph algebra (indeed, it is $A(M)$ where the graph M is displayed above), and she proved that many other finite graph algebras are not finitely based. Eventually, Baker et al. [2,3] characterized the finitely based finite graph algebras. Their characterization is the basis for our duality results for graph algebras. It follows from this characterization and the work of Shallon [26] (see also [17]) that every nonfinitely based finite graph algebra generates a residually large variety. This means that Park's conjecture holds outright for graph algebras.

The literature on duality theory provides an extensive body of results concerning finite algebras belonging to congruence distributive varieties, as well as a growing collection of results concerning various finite groups, rings, and vector spaces. Finite algebras of any of these kinds are known to be finitely based, regardless of whether they are dualizable. To understand the connection between dualizability and the finite basis property, it seems necessary to consider more pathological finite algebras. This led to our effort to characterize dualizable finite graph algebras.

Our investigation of dualizability among finite graph algebras yields a complete characterization which could be framed as a list of six or seven equivalent conditions. Both the nature of these conditions and the structure of our arguments lead us rather to present this characterization in two parts: one part devoted to the negative conditions, and one part devoted to the positive conditions.

Let G be a graph. We say that G is *complete* provided each pair of vertices of G is joined by an edge and each vertex of G has a loop (that is, an edge joining it to itself). We say that G is *bipartite complete* provided the vertices of G can be partitioned into two disjoint nonempty sets (called the *parts of* G) such that any two vertices belonging to different parts are joined by an edge but no edges join vertices that belong to the same part (and, in particular, no vertex has a loop). We say G is a *loose vertex* provided G has exactly one vertex and no edges. G is 4-*transitive* provided for any vertices u, v, w, and z, if edges join u to v, v to w, and w to z, then there is also an edge joining z to u. The vertices need not be distinct; for example, for a triangle to be 4-transitive it must have a loop at each vertex.

Four small graphs play a key role. They are the graphs M, L_3, T, and P_4 displayed below.



M L_3 T P_4

We begin by stating the characterization of the positive conditions.

Theorem 1. *The following statements are equivalent for any finite graph* G:

(i) $A(G)$ *is dualizable.*

(ii) *Each connected component of* G *is either complete, bipartite complete, or a loose vertex.*

(iii) *G is* 4-*transitive.*
(iv) *The equation* $(x \cdot z) \cdot (y \cdot w) \approx (x \cdot y) \cdot (z \cdot w)$ *is true in* $\boldsymbol{A}(G)$.
(v) *The basic operation* $\cdot : \boldsymbol{A}(G)^2 \to \boldsymbol{A}(G)$ *is a homomorphism.*
(vi) $\boldsymbol{A}(G)$ *is finitely based.*

Here is the theorem characterizing the negative conditions.

Theorem 2. *The following statements are equivalent for any finite graph G*:

(i) $\boldsymbol{A}(G)$ *is inherently non-*κ*-dualizable for every cardinal* κ.
(ii) $\boldsymbol{A}(G)$ *is not dualizable.*
(iii) *At least one of* M, L_3, T, *or* P_4 *is an induced subgraph of* G.
(iv) $\boldsymbol{A}(G)$ *is inherently nonfinitely based.*
(v) $\boldsymbol{A}(G)$ *is not finitely based.*

In Theorems 1 and 2, the equivalence of those conditions only involving the finite basis property, the various nonfinite basis properties, and the purely graph theoretic properties, other than 4-transitivity, were established in Baker et al. [2,3]. A graph which fails to be 4-transitive must have an induced subgraph with 4 or fewer vertices that also fails to be 4-transitive. Then it is not hard to see that one of M, L_3, T, or P_4 must be an induced subgraph. Conversely, each of these four graphs exhibits a failure of 4-transitivity which we will make explicit in Section 2. So any graph with one of these as an induced subgraph must also fail to be 4-transitive. In Theorem 1, the useful equivalence of (iv) and (v) with (ii) is easy and no proof is included. Likewise, in Theorem 2 the implication from (i) to (ii) is obvious. Consequently, the characterization will be complete once we establish:

(ii) $\Rightarrow$ (i) *in Theorem* 1: If G is a finite graph in which each connected component is either complete, bipartite complete, or a loose vertex, then $\boldsymbol{A}(G)$ is dualizable.

(iii) $\Rightarrow$ (i) *in Theorem* 2: Each of $\boldsymbol{A}(M), \boldsymbol{A}(L_3), \boldsymbol{A}(T)$, and $\boldsymbol{A}(P_4)$ is inherently non-κ-dualizable.

The latter of these emerges in Section 2 from our broader investigation of inherently nondualizable algebras. The former is accomplished in Section 4. A recent manuscript by Lampe et al. [13] includes the proof that every finitely based finite graph algebra is actually fully dualizable. (This latter notion is fully developed in the monograph of Clark and Davey [4].)

From the above we see that a graph algebra is either dualizable or it is inherently non-κ-dualizable for every cardinal κ. This sharp break is not the general situation. In a recent paper David Clark, Brian Davey, and Jane Pitkethly showed that every unary algebra is embeddable in a dualizable one. So there are nondualizable unary algebras, but there are no inherently nondualizable unary algebras.

The authors would like to thank the referees for a very careful reading of this paper. Their suggestions helped us avoid some embarrassing blunders. We especially appreciate the suggestion that our use of the ghost element method should yield

inherent non-κ-dualizability for each infinite cardinal κ, not just for ω. We also appreciate the suggestion that a more elegant proof of the dualizability of 4-transitive graph algebras might take advantage of the subdirect decomposition of such graph algebras. The structure of the last two sections of this paper owes much to this suggestion.

2. Inherently nondualizable algebras

We employ the ghost element method to establish that certain algebras are not dualizable. This method, which is used implicitly in Davey and Werner [9] to prove that the 2-element implication algebra is not dualizable and more plainly in the proof by Davey et al. of the Big NU Obstacle Theorem [8], emerges from Lemma 1 below. The element g mentioned in part (c) of this lemma is called the *ghost element*. This lemma is, in essence, a reformulation of the failure of an algebra to be κ-dualizable. A proof of this lemma for the $\kappa=\omega$ case can be found in Chapter 10 of the monograph of Clark and Davey [4]. The generalization to arbitrary κ is easy and left to the reader.

Whenever $\boldsymbol{A}$ is a subalgebra of $\boldsymbol{B}^C$, we will let ρ_c denote the *restriction to A of the cth projection*. In particular, this notation is used in the definition of the ghost element in part (c) below.

Lemma 1. *Let $\boldsymbol{B}$ be a finite algebra. Suppose that X is a set, that $\boldsymbol{D}$ is a subalgebra of $\boldsymbol{B}^X$, and that $\alpha\colon \mathrm{Hom}(\boldsymbol{D},\boldsymbol{B})\to B$. Suppose also that*:

(a) *there exists a finite subset C of D such that for any $\sigma,\tau\in\mathrm{Hom}(\boldsymbol{D},\boldsymbol{B})$, if σ and τ agree on C, then $\alpha(\sigma)=\alpha(\tau)$ (that is, α has finite support)*;

(b) *for every subset F of $\mathrm{Hom}(\boldsymbol{D},\boldsymbol{B})$ of cardinality less than κ there exists $d\in D$ such that $\alpha(\sigma)=\sigma(d)$ for each $\sigma\in F$ (that is, α is κ-locally an evaluation)*;

(c) *the element g is defined by $g(x)=\alpha(\rho_x)$, for each $x\in X$.*

If g is in B^X-D, then $\boldsymbol{B}$ is not κ-dualizable.

Invoking this lemma involves constructing α, $\boldsymbol{D}$, and g with the appropriate properties. In many applications, it turns out that an appropriate α can be constructed provided a suitable algebra $\boldsymbol{D}$ and a suitable ghost element g are available. Indeed, in most instances, even g can be constructed from $\boldsymbol{D}$ (or with a similar effort $\boldsymbol{D}$ can be devised once a suitable ghost element g is in hand).

We will use the ghost element method to obtain the stronger conclusion of non-κ-dualizability of certain finite algebras $\boldsymbol{B}$ such that $\boldsymbol{H}$ is a subalgebra of a direct power of $\boldsymbol{B}$. Our plan is to construct $\boldsymbol{D}$ from $\boldsymbol{H}$ and then to focus on the congruences of $\boldsymbol{D}$ which are kernels of homomorphisms into $\boldsymbol{B}$. This leads us to the lemma below.

Let N be a natural number and $S\subseteq T$. We shall say S is a *co-N subset* of T in case $|T-S|<N$.

Lemma 2. *Let $\boldsymbol{H}$ and $\boldsymbol{B}$ be finite algebras with $\boldsymbol{H} \in \boldsymbol{SP}\,\boldsymbol{B}$ and let κ be an infinite cardinal and N be a natural number. Suppose that*:

(a) *Z is a set and $\boldsymbol{D}$ is a subalgebra of $\boldsymbol{H}^Z$, and T is a subset of D of cardinality κ*;

(b) *if Θ is the kernel of a homomorphism from $\boldsymbol{D}$ into $\boldsymbol{B}$, then $\Theta \restriction_T$ has a co-N block*;

(c) *the element g_0 is defined by $g_0(z) = \rho_z(t)$, for each $z \in Z$ and any t from some (chosen) block of maximum size of $(\ker(\rho_z))\restriction_T$.*

The following hold:

(i) *If $\boldsymbol{C}$ is any finite algebra in $\boldsymbol{SP}\,\boldsymbol{B}$, then there is a natural number M so that if Θ is the kernel of any homomorphism from $\boldsymbol{D}$ into $\boldsymbol{C}$ then $\Theta\restriction_T$ has a co-M block. In particular, there is only one block of maximum size of $(\ker(\rho_z))\restriction_T$ in the definition of g_0 above.*

(ii) *If g_0 is in $H^Z - D$, then $\boldsymbol{B}$ is non-κ-dualizable.*

Proof. Let $\boldsymbol{C}$ be any finite algebra in $\boldsymbol{SP}\,\boldsymbol{B}$. For some set I the algebra $\boldsymbol{C}$ is a subalgebra of $\boldsymbol{B}^I$, and without loss of generality, we may suppose I is finite. Let σ be any homomorphism from $\boldsymbol{D}$ into $\boldsymbol{C}$. It is easy to see that

$$\ker(\sigma) = \bigcap_{i\in I} \ker(\rho_i \circ \sigma).$$

For each $t \in T$ we let $[t]_\sigma$ denote the block of t under the relation $\ker(\sigma)\restriction_T$. This equation then yields for any t that

$$[t]_\sigma = \bigcap_{i\in I} [t]_{\rho_i\circ\sigma}.$$

Each $\ker(\rho_i \circ \sigma)\restriction_T$ has a co-N block. Their being cofinite and I being finite allows us to choose t in the intersection of those blocks. Now the latter equation implies that $[t]_\sigma$ is a co-$(((N-1)\cdot|I|)+1)$ block, proving (i).

For some set Y the algebra $\boldsymbol{H}$ is a subalgebra of $\boldsymbol{B}^Y$, and without loss of generality, we may suppose Y is finite since H is finite. Set $X = Z \times Y$. So $\boldsymbol{H}^Z$ is isomorphic to a subalgebra of $\boldsymbol{B}^X$ under an embedding ι such that $\iota(f)(\langle z, y\rangle) = f(z)(y)$. We let D' be the image of D under ι. Hence $\boldsymbol{D}$ is isomorphic to $\boldsymbol{D}'$ which is a subalgebra of $\boldsymbol{B}^X$. We set $T' = \iota(T)$. So $|T'| = \kappa$, and the remaining hypotheses are true for $\boldsymbol{D}'$ and T'.

We define $\alpha : \mathrm{Hom}(\boldsymbol{D}', \boldsymbol{B}) \to B$ by setting $\alpha(\sigma)$ to be the value of σ at any element of the cofinite class of $(\ker(\sigma))\restriction_{T'}$.

Let S be any subset of T' of size $2N-1$. It is not hard to check S is a (finite) support set for α.

T' has cardinality κ which is infinite. So any intersection of fewer than κ cofinite subsets of T' also has cardinality κ and is thus nonempty. So it is easy to check that α is κ-locally an evaluation map.

Now consider g defined by $g(x) = \alpha(\rho_x)$ for each $x \in X$ as in (c) of Lemma 1. Let $t \in T$ and $z \in Z$. Using the notation from above, it is easy to check that

$$\iota([t]_{\rho_z}) = \bigcap_{y\in Y} [\iota(t)]_{\rho_{\langle z,y\rangle}}.$$

It follows that $\iota(g_0) = g$. Since by hypothesis, $g_0 \notin D$, we have $g \notin D'$.
By Lemma 1, $\boldsymbol{B}$ is not κ-dualizable. $\square$

The next theorem is our principal tool for establishing inherent nondualizability. It turns out to be easy to apply.

Theorem 3. *Let $\boldsymbol{H}$ be a finite algebra and κ be an infinite cardinal. Suppose that*:

(a) *Z is a set and $\boldsymbol{D}$ is a subalgebra of $\boldsymbol{H}^Z$, and T is a subset of D of cardinality κ*;
(b) *there is a function u on the natural numbers such that if Θ is a congruence relation on $\boldsymbol{D}$ of finite index at most n then $\Theta\restriction_T$ has at most one class with more than $u(n)$ elements*;
(c) *the element g_0 is defined by $g_0(z) = \rho_z(t)$, for each $z \in Z$ and any t in the block of $(\ker(\rho_z))\restriction_T$ with size greater than $u(|H|)$.*

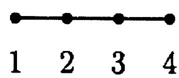
If g_0 is in $H^Z - D$, then $\boldsymbol{H}$ is inherently non-κ-dualizable.

Proof. Let $\boldsymbol{B}$ be any finite algebra with $\boldsymbol{H} \in \boldsymbol{SP}\,\boldsymbol{B}$, and let $n = |B|$, and suppose Θ is the kernel of a homomorphism from $\boldsymbol{D}$ into $\boldsymbol{B}$. Observe that $\Theta\restriction_T$ has one infinite class, and the rest of the classes are of size bounded by $u(n)$. So the complement of the infinite class has fewer than $N = (n-1)u(n) + 1$ members. Lemma 2 implies that $\boldsymbol{B}$ is non-κ-dualizable, and so $\boldsymbol{H}$ is inherently non-κ-dualizable. $\square$

Many applications of the preceding theorem are to situations in which the function u is the constant function with value 1. We wish to use this theorem to prove the next lemma. Observe that if the graph G_0 is an induced subgraph of the graph G_1, then the algebra $\boldsymbol{A}(G_0)$ is a subalgebra of $\boldsymbol{A}(G_1)$. Thus the next lemma establishes the implication (iii) $\Rightarrow$ (i) in Theorem 2.

Lemma 3. *If G is one of the finite graphs M, L_3, P_4, or T and κ is any infinite cardinal, then $\boldsymbol{A}(G)$ is inherently non-κ-dualizable.*

The statement of this lemma suggests that the proof will be a four-case argument. By abstracting, we can get it down to one case. We begin by considering P_4 and the structure of $\boldsymbol{A}(P_4)$. We label P_4 as indicated in the figure below.



Recall that $x \cdot y = 0$ in $\boldsymbol{A}(G)$ unless x and y are joined by an edge of G, and in that case, $x \cdot y = x$. The reader can easily check that in $\boldsymbol{A}(P_4)$

$$0 = x \cdot 0 = 0 \cdot x = 4 \cdot 1, \quad 2 = 2 \cdot 1 = 2 \cdot 3, \quad 3 = 3 \cdot 2 = 3 \cdot 4 \quad \text{and} \quad 4 = 4 \cdot 3.$$

In other words, the operation table for $\boldsymbol{A}(P_4)$ is partly filled in as follows:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0				
2	0	2		2	
3	0		3		3
4	0	0		4	

We say that the algebra $\boldsymbol{H} = \langle H, \cdot \rangle$ is P_4-*like* provided

(i) H has a subset H_0 which consists of at most 5 elements named 0, 1, 2, 3, 4,
(ii) $3 \neq 4$,
(iii) $x \cdot y = 4$ implies $x = 4$ for all $x, y \in H$, and
(iv) in $\boldsymbol{H}$ the operation $\cdot$ obeys the stipulations indicated in the above partial operation table.

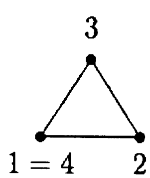
We specifically do *not* require that the set H_0 above have five distinct elements, but H_0 has to have at least two elements since $3 \neq 4$. Suppose that $0 = 4$. Then we would have $3 = 3 \cdot 4 = 3 \cdot 0 = 0 = 4$, but $3 \neq 4$. Similarly, the cases $3 = 0$, $2 = 0$, and $1 = 0$ each imply $3 = 4$. Since $3 \neq 4$, it is the case that $0 \notin \{1, 2, 3, 4\}$. So H_0 can only be a set of size 3, 4, or 5. In addition

$$2 \neq 4 \neq 3.$$

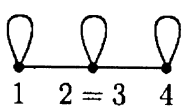
Indeed suppose $2 = 4$. We would then have $0 = 4 \cdot 1 = 2 \cdot 1 = 2$, but $0 \neq 2$.

What is also important for our purposes is that our four graph algebras are P_4-like. Certainly $\boldsymbol{A}(P_4)$ is P_4-like. The remaining three are shown to be P_4-like by the following diagrams and tables (or by considering the following diagrams and the definition of $\cdot$ for graph algebras).

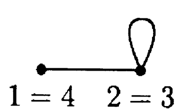
T



L_3



M



$\mathbf{A}(T)$

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	1	1
2	0	2	0	2
3	0	3	3	0

$\mathbf{A}(L_3)$

$\cdot$	0	1	2	4
0	0	0	0	0
1	0	1	1	0
2	0	2	2	2
4	0	0	4	4

$\mathbf{A}(M)$

$\cdot$	0	1	2
0	0	0	0
1	0	0	1
2	0	2	2

Incidentally, the labeling of T, L_3, and M given above make it clear why each of these graphs fails to be 4-transitive. Obviously, P_4 fails 4-transitivity.

Lemma 4. *If $\mathbf{H}$ is P_4-like and κ is any infinite cardinal, then $\mathbf{H}$ is inherently non-κ-dualizable.*

Lemma 3 now follows immediately from the fact that all four of the graph algebras in question are P_4-like.

Proof. We will employ Theorem 3 in this proof.

Some notation proves useful. Suppose x and $y_1,\ldots,y_k$ are objects and $\alpha_1,\ldots,\alpha_k$ are ordinals less than κ such that $y_i = y_j$ whenever $\alpha_i = \alpha_j$. Then we let the sequence

$$x^{y_1,\ldots,y_k}_{\alpha_1,\ldots,\alpha_k} = \langle z_\beta\colon \beta < \kappa\rangle,$$

where $z_{\alpha_j} = y_j$, for $1 \leqslant j \leqslant k$, and $z_\beta = x$ otherwise. For example, $4^{2,0}_{3,7}$ denotes the sequence of length κ

$$4, 4, 4, 2, 4, 4, 4, 0, 4, 4, 4, 4, \ldots,$$

where the 2 sits at coordinate 3, and the 0 sits at coordinate 7, and 4 sits at every other coordinate.

Suppose $\mathbf{H}$ is P_4-like. Let

$$T = \{4^2_\alpha\colon \alpha < \kappa\}.$$

We let $\mathbf{D}$ be the subalgebra of $\mathbf{H}^\kappa$ generated by

$$\{3^1_\alpha\colon \alpha < \kappa\} \cup T \cup \{\mathbf{3}\},$$

where $\mathbf{3}$ is the constant sequence in H^κ with value 3.

The following equations all hold in $\mathbf{D}$ as can be verified using the definition of P_4-like.

$$\begin{aligned}
&4^2_\alpha = 4^2_\alpha \cdot (\mathbf{3} \cdot 4^2_\beta),\\
&4^2_\alpha = 4^2_\alpha \cdot 3^1_\alpha,\\
&4^{2,0}_{\beta,\alpha} = 4^2_\beta \cdot 3^1_\alpha,\\
&4^{0,0}_{\beta,\alpha} = 4^{2,0}_{\beta,\alpha} \cdot (\mathbf{3} \cdot 4^{0,2}_{\beta,\alpha}),\\
&4^{0,0,0,0}_{\delta,\gamma,\beta,\alpha} = 4^{0,0}_{\delta,\gamma} \cdot (\mathbf{3} \cdot 4^{0,0}_{\beta,\alpha}).
\end{aligned}$$

Let $\Theta \in \mathrm{Con}(\mathbf{D})$. Suppose $\alpha \neq \beta$ and $\gamma \neq \delta$ and $4^2_\alpha \equiv 4^2_\beta \pmod{\Theta}$ and $4^2_\gamma \equiv 4^2_\delta \pmod{\Theta}$. Then also we have

$$4^2_\alpha = 4^2_\alpha \cdot 3^1_\alpha \equiv 4^2_\beta \cdot 3^1_\alpha = 4^{2,0}_{\beta,\alpha} \pmod{\Theta}.$$

Symmetrically we have that $4^2_\beta \equiv 4^{0,2}_{\beta,\alpha} \pmod{\Theta}$. It follows that

$$4^2_\alpha = 4^2_\alpha \cdot (\mathbf{3} \cdot 4^2_\beta) \equiv 4^{2,0}_{\beta,\alpha} \cdot (\mathbf{3} \cdot 4^{0,2}_{\beta,\alpha}) = 4^{0,0}_{\beta,\alpha} \pmod{\Theta}.$$

Thus also $4^2_\gamma \equiv 4^{0,0}_{\delta,\gamma} \pmod{\Theta}$. Now we have

$$4^2_\alpha = 4^2_\alpha \cdot (\mathbf{3} \cdot 4^2_\gamma) \equiv 4^{0,0}_{\beta,\alpha} \cdot (\mathbf{3} \cdot 4^{0,0}_{\delta,\gamma}) = 4^{0,0,0,0}_{\delta,\gamma,\beta,\alpha} \pmod{\Theta}.$$

Symmetrically we have $4^2_\gamma \equiv 4^{0,0,0,0}_{\delta,\gamma,\beta,\alpha} \pmod{\Theta}$, and so $4^2_\alpha \equiv 4^2_\gamma \pmod{\Theta}$. Thus $\Theta\restriction_T$ has at most one block with more than one element. Therefore, (b) of the hypotheses of Theorem 3 is established with $u(n) = 1$ for all n.

Recall that ρ_α denotes the restriction of the αth projection function to D. For any $\beta \neq \alpha$, we have $\rho_\alpha(4^2_\beta)=4$. So the element g_0 of H^κ considered in (c) of the hypotheses of Theorem 3 is $\mathbf{4}$, the constant sequence with value 4. Since $2 \neq 4 \neq 3$, the sequence $\mathbf{4}$ was not in the generating set of $\boldsymbol{D}$, and it follows from item (iii) of the definition of 'P_4-like' that $g_0 = \mathbf{4} \notin D$. So an application of Theorem 3 finishes the proof of this lemma. $\square$

The stipulations that make up the definition of 'P_4-like' were of two kinds. Those imposed by the conditions in the partial operation table ensured that $\Theta\restriction_T$ had at most one nontrivial block for each $\Theta \in \mathrm{Con}(\boldsymbol{D})$. The stipulations (ii) and (iii) of the definition of 'P_4-like', on the other hand, were needed to exclude the ghost element from D.

We can apply the same ideas to algebras with just two elements. We say that the algebra $\boldsymbol{H}$ is *implication-like* provided

(i) $H = \{0, 1\}$,
(ii) $0 \neq 1$,
(iii) for each fundamental operation F of $\boldsymbol{H}$ there is an index k less than the rank r of F such that $F(a_0, a_1, \ldots, a_{r-1}) = 0$ implies $a_k = 0$, for all $a_0, a_1, \ldots, a_{r-1} \in H$, and
(iv) $x \vee (y \wedge z)$ is a term function of $\boldsymbol{H}$, where $\vee$ and $\wedge$ take their customary meanings as operations on the 2-element lattice.

Our next lemma improves Theorem 10.5.3 of Clark and Davey [4] by strengthening the conclusion from nondualizability to inherent nondualizability.

Lemma 5. *Every implication-like algebra is inherently non-κ-dualizable, for every infinite cardinal κ.*

Proof. We use the notation developed in the proof of Lemma 4. Take $T = \{0^1_\alpha \colon \alpha < \kappa\}$ and let $\boldsymbol{D}$ be the subalgebra of $\boldsymbol{H}^\kappa$ generated by T.

Now, suppose $\alpha, \beta, \gamma, \delta < \kappa$ with $\alpha \neq \beta$ and $\gamma \neq \delta$. Suppose also that $\Theta \in \mathrm{Con}(\boldsymbol{D})$ and $0^1_\alpha \equiv 0^1_\beta \pmod{\Theta}$ and $0^1_\gamma \equiv 0^1_\delta \pmod{\Theta}$. Then modulo Θ we also have

$$\begin{aligned} 0^1_\alpha &= 0^1_\alpha \vee (0^1_\delta \wedge 0^1_\gamma) \\ &\equiv 0^1_\beta \vee (0^1_\delta \wedge 0^1_\delta) \\ &= 0^{1,1}_{\beta,\delta} \end{aligned}$$

$$
\begin{aligned}
&= 0^{1,1}_{\delta,\beta} \\
&= 0^1_\delta \vee (0^1_\beta \wedge 0^1_\beta) \\
&\equiv 0^1_\gamma \vee (0^1_\beta \wedge 0^1_\alpha) = 0^1_\gamma.
\end{aligned}
$$

Thus $\Theta \restriction_T$ has at most one block with more than one element, for any $\Theta \in \mathrm{Con}(\boldsymbol{D})$. Therefore, (b) of the hypotheses of Theorem 3 is established with $u(n) = 1$ for all n.

Recall that ρ_β denotes the restriction of the βth projection function to D. For any $\alpha \neq \beta$, we have $\rho_\beta(0^1_\alpha) = 0$. So the element g_0 of H^κ considered in (c) of the hypotheses of Theorem 3 is $\mathbf{0}$, the constant sequence with value 0. Notice that from item (ii) of the definition of 'implication-like' $\mathbf{0}$ is not in T, the generating set of $\boldsymbol{D}$. It follows from item (iii) of the definition of 'implication-like' that $g_0 = \mathbf{0} \notin D$. So an application of Theorem 3 finishes the proof of this lemma. □

Up to term equivalence, there are just eight two-element algebras which are implication-like. All of them are inherently nondualizable. Included among them is the implication algebra displayed below:

$\to$	0	1
0	1	1
1	0	1

In the implication algebra $x \vee (y \wedge z)$ is given by the term $(y \to (z \to x)) \to x$. Seven further algebras on $\{0, 1\}$ which represent the other term-equivalence classes of inherently nondualizable two element algebras are elaborated in Chapter 10 of Clark and Davey [4], where proofs of the dualizability of all other two-element algebras can also be found.

Since the implication algebra is a two-element groupoid, we obtain the following corollary by crudely counting the groupoids into which the implication algebra can be embedded.

Corollary 6. *There is a constant c such that for every natural number $n > 1$, there are at least cn^{n^2-2} groupoids on the set $\{0, 1, \ldots, n-1\}$ which are inherently non-κ-dualizable for every infinite cardinal κ.*

Proof. Consider the operation table of an n-element groupoid as a square array with n^2 cells. We will count just some of the ways in which these cells can be assigned values from $\{0, 1, 2, \ldots, n-1\}$ so that the implication algebra or its opposite, the tables for both displayed below,

$\to$	0	1
0	1	1
1	0	1

$\leftarrow$	0	1
0	1	0
1	1	1

will be embedded in the resulting array. To guard against counting arrays more than once, we will use the following properties of these two algebras: each has exactly one idempotent element and this idempotent element has exactly two ‘square roots’. In the $n \times n$-arrays we will count, we impose these two properties as well. This will ensure that no array is counted twice.

We have n choices for the idempotent element and $n-1$ choices for the second ‘square root’ of the idempotent element. These choices amount to filling two of the cells on the diagonal of our array. There remain $n-2$ cells along the diagonal of our array. We are constrained not to fill them with the idempotent we have chosen, lest our chosen idempotent have more than two ‘square roots’. We are also constrained not to fill them in such a way as to create a second idempotent element. Thus for each of these $n-2$ cells we have $n-2$ choices. Now we are also forced to fill two of the off-diagonal cells so as to obtain the desired embedding of the implication algebra or its opposite. There are 2 ways to do this. For each of the remaining $n^2-(n+2)$ off-diagonal cells we have n choices. Altogether, this means that there are at least

$$n(n-1)(n-2)^{(n-2)}2n^{n^2-n-2}$$

groupoids on $\{0,1,2,\dots,n-1\}$ into which the implication algebra or its opposite can be embedded.

For this argument we will take $c=2\cdot 3^{-3}$, although larger constants are possible. Now some simple calculations show that

$$2n(n-1)(n-2)^{(n-2)}n^{n^2-n-2}\geqslant 2\cdot 3^{-3}n^{n^2-2},$$

if and only if

$$\frac{n(n-1)}{(n-2)^2}\left(\frac{n-2}{n}\right)^n\geqslant 3^{-3}.$$

But now note that when $n>2$ we have

$$\frac{n(n-1)}{(n-2)^2}\left(\frac{n-2}{n}\right)^n\geqslant\left(\frac{n-2}{n}\right)^n\geqslant 3^{-3}.$$

Indeed, the quantity $((n-2)/n)^n$ in the last inequality is easily shown to be increasing to the limit e^{-2}. So, asymptotically 3^{-3} should be replaced by e^{-2}. More careful counting leads to still larger constants. For $n=2$, this corollary is obvious. □

This result stands in contrast to the discovery of Murskiĭ [20] that almost all finite groupoids are idemprimal (that is, every finitary idempotent function is a term function). Since Davey and Werner [9] proved that every finite algebra with a near-unanimity term is dualizable, it follows that almost all finite algebras are dualizable. Sharp estimates on how quickly the proportion of nondualizable groupoids on an n-element set drops toward 0 are unknown, but it certainly cannot be quicker than on the order of n^{-2}.

In Heindorf [11] it was proved that there are 2^{ω} nondualizable clones on a set with three elements. On the other hand, Porebska [23] showed how to construct 2^{ω} clones on a set with three elements, each generating a congruence distributive 3-permutable variety. Following the reasoning of Idziak in [12], as a second corollary to Lemma 5 we can obtain a stronger combined form of these results.

Corollary 7. *There are 2^{ω} clones on $\{0,1,2\}$ each of which is inherently non-κ-dualizable for every infinite cardinal κ and each of which generates a congruence distributive, congruence 3-permutable variety.*

Proof. Define the binary operation $\to$ and the n-ary operation F_n on $\{0,1,2\}$ as follows:

$\to$	0	1	2
0	1	1	2
1	0	1	2
2	0	1	1

and

$$F_n(x_0,x_1,\ldots,x_{n-1}) = \begin{cases} 2 & \text{if } x_0 = x_1 = \cdots = x_{n-1} = 2, \\ 0 & \text{if } x_0,x_1,\ldots,x_{n-1} \in \{0,2\} \text{ with } x_k = 2 \\ & \quad \text{for exactly one } k, \\ 1 & \text{otherwise.} \end{cases}$$

Now let S be any set of natural numbers larger than 2 and take $\boldsymbol{B}_S$ to be the algebra with universe $\{0,1,2\}$ and fundamental operations $\to$ and F_n for all $n \in S$. Let $\boldsymbol{H}_S$ denote the subalgebra with universe $\{0,1\}$, (which is easily seen to be a subuniverse of $\boldsymbol{B}_S$). Plainly, $\boldsymbol{H}_S$ is implication-like. Consequently, $\boldsymbol{B}_S$ is inherently non-κ-dualizable.

As Idziak pointed out in [12], (an isomorphic copy of) $\boldsymbol{B}_\emptyset$ is embeddable into the direct square of the two-element implication algebra, and therefore each $\boldsymbol{B}_S$ generates a congruence distributive, congruence 3-permutable variety according to Mitschke [18]. That these algebras all generate distinct clones was proven by Idziak in [12]. □

The Big NU Obstacle Theorem asserts, in part, that a finite algebra $\boldsymbol{H}$ which has no near unanimity terms but which generates a quasivariety all of whose finite algebras are (relatively) congruence join-semidistributive must fail to be dualizable. The same strategy used in the arguments above is at work in [8] in the proof of the Big NU Obstacle Theorem. Those arguments actually yield the following strengthened version.

The Inherent NU Obstacle Theorem (Davey et al. [8], Clark and Davey [4]). *Let $\boldsymbol{H}$ be a finite algebra which has no near unanimity term.*

(i) *If every finite algebra in $\mathbb{SP}\boldsymbol{H}$ is congruence join-semidistributive, then $\boldsymbol{H}$ is* inherently *non-κ-dualizable, for every infinite cardinal κ.*

(ii) *If $\boldsymbol{B}$ is a finite algebra such that $\boldsymbol{H} \in \mathbb{SP}\boldsymbol{B}$ and every finite algebra in $\mathbb{SP}\boldsymbol{H}$ is congruence join-semidistributive* relative *to $\mathbb{SP}\boldsymbol{B}$, then $\boldsymbol{B}$ is non-κ-dualizable, for every infinite cardinal κ.*

To show the unity of the ideas, we will give the proof of this theorem. McKenzie's original (unpublished) proof of the Big NU Obstacle Theorem used a ghost element argument, but the published proof did not. However, the heart of the following argument is lifted, with only slight notational changes, directly from pp. 437–438 of [8].

Proof. First we consider (i). Let $\boldsymbol{F}$ be the algebra freely generated by x and y in the quasivariety generated by $\boldsymbol{H}$. Take the κ direct power of $\boldsymbol{F}$ and set

$$T = \{x_\alpha^y\colon\ \alpha \in \kappa\}.$$

Let $\boldsymbol{D}$ be the subalgebra generated by T. Now let Θ be any congruence of $\boldsymbol{D}$ of finite index. We want to show that $\Theta\restriction_T$ has one big block and a bunch of singletons. So pick distinct α, β, γ, and δ less than κ. Suppose that

$$x_\alpha^y \equiv x_\beta^y \,(\operatorname{mod} \Theta) \quad \text{and} \quad x_\gamma^y \equiv x_\delta^y \,(\operatorname{mod} \Theta).$$

Let $\boldsymbol{C}$ be the subalgebra generated by the four elements $x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y$, and let Θ_0 be the restriction of Θ to C.

Every element of C can be written as the value, $t(x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y)$, of some 4-ary term t in the language of $\boldsymbol{H}$. For each $\varepsilon < \kappa$ we have that $t(x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y)(\varepsilon)$ is an element of F and is either $t(y,x,x,x), t(x,y,x,x), t(x,x,y,x), t(x,x,x,y)$, or $t(x,x,x,x)$ depending on whether ε is $\alpha, \beta, \gamma, \delta$, or some other ordinal less than κ. Since $\boldsymbol{F}$ is the free algebra on two generators, for each $\varepsilon < \kappa$ and each pair of terms, we have that the following implication is true throughout the quasivariety generated by $\boldsymbol{H}$.

$$t_1(x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y)(\varepsilon) = t_2(x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y)(\varepsilon) \Rightarrow t_1(x,x,x,x) = t_2(x,x,x,x). \tag{1}$$

For any $\varepsilon < \kappa$ we let Π_ε be the kernel of the projection homomorphism of $\boldsymbol{C}$ onto $\boldsymbol{F}$ at coordinate ε. Let ε be any ordinal less than κ which is not in $\{\alpha, \beta, \gamma, \delta\}$. One consequence of implication (1) is that $\Pi_\alpha \leqslant \Pi_\varepsilon$. Similar inequalities hold with β, γ, and δ taking the place of α. It follows that

$$\Pi_\alpha \wedge \Pi_\beta \wedge \Pi_\gamma \wedge \Pi_\delta = \bigwedge (\Pi_\varepsilon\colon\ \varepsilon < \kappa),$$

which is the identity relation on C. Therefore,

$$\Theta_0 = \Theta_0 \vee (\Pi_\alpha \wedge \Pi_\beta \wedge \Pi_\gamma \wedge \Pi_\delta). \tag{2}$$

Now let $\varepsilon < \kappa$ and suppose $t_1(x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y) \equiv t_2(x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y) \,(\operatorname{mod} \Pi_\varepsilon)$. It follows from implication (1) that the identity $t_1(x,x,x,x) = t_2(x,x,x,x)$ is true throughout the quasivariety. Then we can conclude that

$$\begin{aligned}
t_1(x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y) &\equiv t_1(x_\alpha^y, x_\beta^y, x_\beta^y, x_\beta^y) \,(\operatorname{mod} \Pi_\alpha)\\
&\equiv t_1(x_\alpha^y, x_\alpha^y, x_\alpha^y, x_\alpha^y) \,(\operatorname{mod} \Theta_0)\\
&= t_2(x_\alpha^y, x_\alpha^y, x_\alpha^y, x_\alpha^y)\\
&\equiv t_2(x_\alpha^y, x_\beta^y, x_\beta^y, x_\beta^y) \,(\operatorname{mod} \Theta_0)\\
&\equiv t_2(x_\alpha^y, x_\beta^y, x_\gamma^y, x_\delta^y) \,(\operatorname{mod} \Pi_\alpha).
\end{aligned}$$

Thus we have that $\Pi_\varepsilon \leqslant \Theta_0 \vee \Pi_\alpha$. Similar inequalities hold with β, γ, and δ taking the place of α. Since ε was arbitrary, it follows that

$$\Theta_0 \vee \Pi_\alpha = \Theta_0 \vee \Pi_\beta = \Theta_0 \vee \Pi_\gamma = \Theta_0 \vee \Pi_\delta.$$

Now Eq. (2) and the congruence join-semidistributivity of $\boldsymbol{C}$ imply that

$$\Theta_0 = \Theta_0 \vee (\Pi_\alpha \wedge \Pi_\beta \wedge \Pi_\gamma \wedge \Pi_\delta) = \Theta_0 \vee \Pi_\alpha.$$

That is, $\Pi_\alpha \leqslant \Theta_0$, and so $x^y_\beta \equiv x^y_\gamma \pmod{\Theta_0}$. Hence all of $x^y_\alpha, x^y_\beta, x^y_\gamma$, and x^y_δ are related by Θ and $\Theta\restriction_T$ has only one big block.

Now the ghost element is just the constantly x sequence. Suppose it is in D. This means there is a term $t(x_0, \ldots, x_{r-1})$ so that evaluating this term in D at an appropriate choice of the generators (the elements of T) gives the constantly x sequence. But remembering how D is a subalgebra of the κ direct power of the free algebra and thinking about things coordinatewise, we see that the near unanimity equations are easily derivable from this supposition.

Thus, if there are no near unanimity terms then there is no way to get that constantly x tuple into D. This means it qualifies as a ghost element. Theorem 3 finishes the proof of (i).

Part (ii) is proved in almost exactly the same way except that we apply Lemma 2 instead of Theorem 3. $\square$

The technique used in this section to invoke the ghost element method involved defining a map α on the set of congruences of finite index. In essence, this α selected the 'large' block of the congruence. Moreover, item (b) in Lemma 1 amounts to the stipulation that the collection of these 'large' blocks has the λ-intersection property for each $\lambda < \kappa$. So there is a filter implicit in our technique (and in the application to P_4-like algebras this filter was just the Frechet filter — the filter of cofinite subsets of T).

Let T be any set, $\mathscr{F}$ be a proper filter on T, and $\mathscr{C}$ be a family of equivalence relations on T of finite index. We say that $\mathscr{F}$ is a *prime filter relative to* $\mathscr{C}$ provided each equivalence relation Θ in $\mathscr{C}$ has a (necessarily unique) block B_Θ which belongs to $\mathscr{F}$. We call such a block *$\mathscr{F}$-big*. If, in addition, the intersection of fewer than κ members of $\mathscr{F}$ always belongs to $\mathscr{F}$ we say that $\mathscr{F}$ is *$<\kappa$-complete*. We say $\mathscr{F}$ *has finitely bounded support* provided for every natural number n there is a finite set $C_n \subseteq T$ such that for $\Theta, \Phi \in \mathscr{C}$ each of index at most n if $\Theta\restriction_{C_n} = \Phi\restriction_{C_n}$, then $C_n \cap B_\Theta \cap B_\Phi$ is not empty.

We can now rewrite Theorem 3 in the language of filters. The proof of Lemma 2 can be adapted to this setting.

Lemma 8. *Let $\boldsymbol{H}$ be a finite algebra and κ be an infinite cardinal. Suppose that*

(a) *Z is a set and $\boldsymbol{D}$ is a subalgebra of $\boldsymbol{H}^Z$, and T is an infinite subset of D*;

(b) *there is a filter $\mathscr{F}$ on T prime relative to $\mathscr{C}=\{\Theta\restriction_T\colon \Theta \in \operatorname{Con}\boldsymbol{D}$ of finite index$\}$ which has finitely bounded support and is $<\kappa$-complete*;

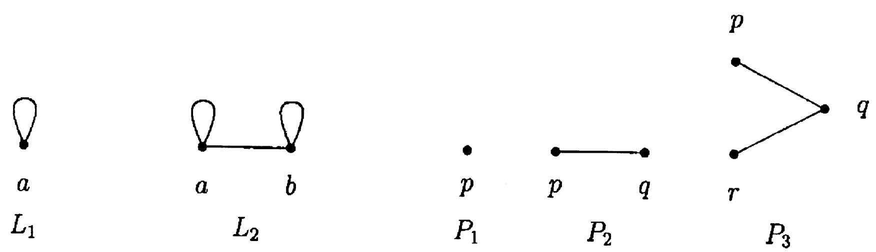
(c) *the element* g_0 *is defined by* $g_0(z) = \rho_z(t)$ *for each* $z \in Z$ *and for any* t *in the* $\mathscr{F}$*-big block of* $(\ker(\rho_z))\restriction_T$.

If g_0 *is in* $H^Z - D$, *then* $\boldsymbol{H}$ *is inherently non-*κ*-dualizable.*

A proper filter $\mathscr{F}$ is an *ultrafilter* if it is prime relative to the family of *all* equivalence relations on T of finite index. A proper filter on a set T is *measurable* provided it is $<|T|$-complete. Every filter on a countably infinite set is measurable. The Frechet filter on any infinite set is also measurable. Uncountable sets T which can support a measurable ultrafilter must be enormous, if they exist at all: $|T|$ must be a measurable cardinal. The notion of measurable cardinal can be found in [29] and an account of their enormous magnitude is given in [28]. The existence of uncountable measurable cardinals cannot be deduced from the usual axioms of set theory. For example, Scott [25] showed that the existence of an uncountable measurable cardinal conflicts with Gödel's axiom of constructibility. On the other hand, the filters which interest us here need only be prime relative to certain equivalence relations of finite index, so they can be quite different from ultrafilters. For instance, the Frechet filter on an uncountable set T is measurable and it is just this kind of filter which we have employed in this section.

3. Alter egos of essential 4 -transitive graph algebras

The rest of this paper is devoted to establishing the dualizability of every finite graph algebra in which each connected component is either complete, bipartite complete, or a loose vertex. There are, of course, infinitely many such graphs, but we need not consider all of them. Twelve of them turn out to play essential roles. We describe these 12 graphs next. Each of our 12 graphs will be a disjoint union of at most two of the components displayed and named below.



The two looped components The three loopless components

If G and H are graphs, we denote their disjoint union by $G + H$. We refer to the following 12 graphs as *essential* 4-*transitive graphs*:

$$\emptyset,\ P_1,\ P_2,\ P_3,\ L_1,\ L_2,\ P_1 + L_1,\ P_1 + L_2,\ P_2 + L_1,\ P_2 + L_2,\ P_3 + L_1,\ P_3 + L_2.$$

Likewise we say that the graph algebras of these are the *essential* 4-*transitive graph algebras*.

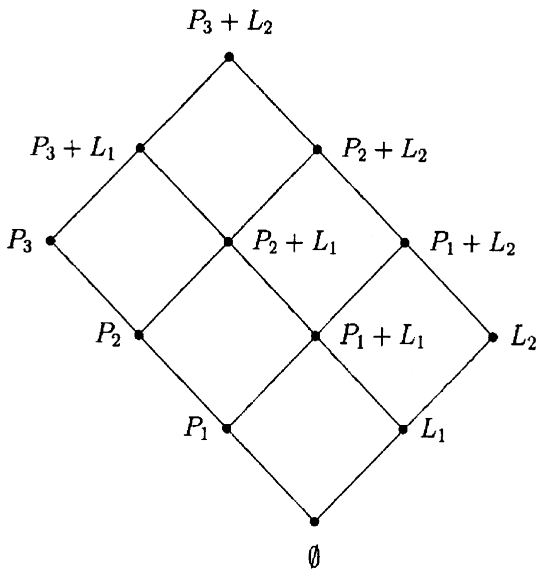
Lemma 9. *If every essential* 4-*transitive graph algebra is dualizable, then every finite* 4-*transitive graph algebra is dualizable. Moreover, every quasivariety generated by a finite* 4-*transitive graph algebra is, in fact generated by an essential* 4-*transitive graph algebra.*

Proof. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be any algebras. We say that $\boldsymbol{B}$ is a *point separating retract* of $\boldsymbol{A}$ provided $\boldsymbol{B}$ is a retract of $\boldsymbol{A}$ (this means that there are homomorphisms $\alpha : \boldsymbol{A} \to \boldsymbol{B}$ and $\beta : \boldsymbol{B} \to \boldsymbol{A}$ so that $\alpha \circ \beta$ is the identity map on $\boldsymbol{B}$) and $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$ separates the points of A (which means that if $a, a' \in A$ with $a \neq a'$, then $f(a) \neq f(a')$ for some $f \in \mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$).

Davey in [7] proved that if $\boldsymbol{A}$ is a finite algebra and $\boldsymbol{B}$ is a dualizable algebra which is a point separating retract of $\boldsymbol{A}$, then $\boldsymbol{A}$ is dualizable.

Thus, what we need to show is that if G is a finite 4-transitive graph, then there is an essential 4-transitive graph H such that $\boldsymbol{A}(H)$ is a point separating retract of $\boldsymbol{A}(G)$.

Ordering the 12 essential 4-transitive graphs by the induced subgraph relation, we arrive at the partially *ordered set of essential* 4-*transitive graphs* shown below.



The ordered set of essential 4-transitive graphs

Now take G to be any finite 4-transitive graph. This means the connected components of G are either loose vertices, complete, or bipartite complete. Thus at least one of the essential 4-transitive graphs is isomorphic to an induced subgraph of G. Let H be the one occurring highest in the ordered set of essential 4-transitive graphs. It is not hard to see that $\boldsymbol{A}(H)$ is a retract of $\boldsymbol{A}(G)$ and that there are enough homomorphisms in $\mathrm{Hom}(\boldsymbol{A}(G), \boldsymbol{A}(H))$ to separate points. Thus $\boldsymbol{A}(G)$ is dualizable. This is also a consequence of the recent theorem of Davey and Willard [10] according to which if two finite algebras generate the same quasivariety and one of them is dualizable, then other must be dualizable as well. Here $\boldsymbol{A}(G)$ and $\boldsymbol{A}(H)$ generate the same quasivariety. □

Now among these 12 graphs, one is the empty graph, five consist of a single component with no more than three vertices, and the remaining six consist of two components where one is looped while the other is loopless.

The work of this section is to describe an alter ego $\mathbb{A}(G)$ of each graph algebra $\boldsymbol{A}(G)$ where G is one of the 12 essential 4-transitive graphs. In the next section we will prove that these alter egos actually impose the dualities we desire. In defining $\mathbb{A}(G)$ we will use the vertex labeling explicitly given above.

We select a as the *designated* vertex of any L_i component, and we select p as the *designated* vertex of any P_i component. The *left part* of P_2 is $\{p\}$, and of P_3 is $\{p,r\}$. The *right part* of either P_2 or P_3 is $\{q\}$. A *left vertex* is one that belongs to a left part. q is the unique *right vertex*.

We will use the following operations, relations, and partial operations to obtain alter egos of our 12 essential 4-transitive graph algebras.

Total functions

Constants: One for each idempotent element — that is, 0 and each of a and b when they are in G.

Unary operations:

f: A permutation f such that

(i) $f(a)=b$ and $f(b)=a$ when both a and b are in G;
(ii) $f(p)=r$ and $f(r)=p$ when both p and r are in G;
(iii) $f(x)=x$ in all other cases and for all other elements.

h: The function such that $h(p)=h(r)=q$, and $h(q)=p$ which fixes every other element.

k_D: For each component D define

$$k_D(u)=\begin{cases} u \text{ if } u\in D,\\ 0 \text{ if } u\notin D.\end{cases}$$

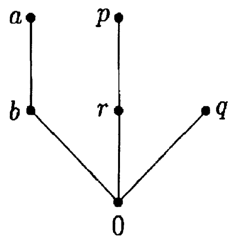
Binary operations:

$\cdot$: The graph operation.

$\wedge$: Called the meet, this operation is defined so that $x\wedge x=x$ for all x, $p\wedge r=r\wedge p=r$, $a\wedge b=b\wedge a=b$, and all other meets give the value 0. The meet operation is easily seen to be a semilattice operation. (The semilattice as an ordered set is drawn below.)

$*$: This operation is defined by

$$u*v=\begin{cases} u & \text{if } u \text{ is a loopless vertex and } v \text{ is a looped vertex},\\ 0 & \text{otherwise}.\end{cases}$$



The semilattice on $P_3 + L_2$

Relations

Unary relations:
$(D - \{e\}) \cup \{0\}$: where D is any component and e is the designated vertex of D.
Binary relations:
R_{LP} : Here L is a looped component and P is a loopless component. Then

$$R_{LP} = [(L \cup \{0\}) \times (P \cup \{0\})] - \left\{ \begin{pmatrix} a \\ p \end{pmatrix} \right\}.$$

Partial binary operations

$\downarrow_p^v$: where v is any element (even 0), this function is defined so that

$$x \downarrow_p^v y = \begin{cases} p & \text{if } x = p \text{ and } y = v, \\ 0 & \text{if } x = p \text{ and } y \neq v, \\ 0 & \text{if } x = 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

That is $p \downarrow_p^v v = p$ and $p \downarrow_p^v y = 0$ when $y \neq v$ and $0 \downarrow_p^v y = 0$ for any y, and otherwise $x \downarrow_p^v y$ is undefined.

It is a routine and tedious matter to check that each of these operations, relations, and partial operations is algebraic over $\boldsymbol{A}(P_3 + L_2)$. We leave these details to the reader. We use the fact that $\wedge$ is a semilattice operation without further explicit mention in forming expressions like $\bigwedge_{k \in I} u_k$ where I can be any nonempty finite index set. Note that each designated element is a maximal element of the semilattice, and q, the right vertex, is a maximal element as well. We will exploit this maximality.

We take the alter ego of $\boldsymbol{A}(G)$ to be the structure with universe $V \cup \{0\}$ (that is, the universe of $\boldsymbol{A}(G)$) endowed with all the restrictions of the operations, relations, and the partial operations listed above, and h is included if and only if both $p, q \in G$, and the partial operations are included if and only if $p \in G$. The alter ego of $\boldsymbol{A}(G)$ is denoted by $\mathbb{A}(G)$. Here is an example.

The structure of $\mathbb{A}(P_2 + L_2)$

Three constants: $0, a$, and b.
Four unary operations:

$$f(x) = \begin{cases} b & \text{if } x = a, \\ a & \text{if } x = b, \\ x & \text{otherwise,} \end{cases} \qquad h(x) = \begin{cases} d & \text{if } x = c, \\ c & \text{if } x = d, \\ x & \text{otherwise,} \end{cases}$$

$$k_{P_2}(u) = \begin{cases} u & \text{if } u \in P_2, \\ 0 & \text{if } u \notin P_2, \end{cases} \qquad k_{L_2}(u) = \begin{cases} u & \text{if } u \in L_2, \\ 0 & \text{if } u \notin L_2. \end{cases}$$

Three binary operations: The graph operation $\cdot$, the meet operation $\wedge$ defined so that $x \wedge x = x$ for all $x, a \wedge b = b \wedge a = b$, while all other meets give the value 0, and

$$u * v = \begin{cases} u & \text{if } u \text{ is a loopless vertex and } v \text{ is a looped vertex,} \\ 0 & \text{otherwise.} \end{cases}$$

Two unary relations: $\{b, 0\}$ and $\{q, 0\}$.
One binary relation: $R = R_{L_2P_2} = [(L_2 \cup \{0\}) \times (P_2 \cup \{0\})] - \left\{ \binom{a}{p} \right\}$.
Five partial binary operations:

$$x \downarrow_p^v y = \begin{cases} p & \text{if } x = p \text{ and } y = v, \\ 0 & \text{if } x = p \text{ and } y \neq v, \\ 0 & \text{if } x = 0, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

where v is any element, even 0.

4. The interpolation condition for essential 4-transitive graph algebras

We will invoke the second duality theorem and the duality compactness theorem to demonstrate that each of our 12 essential 4-transitive graph algebras is dualizable.

Suppose $\boldsymbol{A}$ is an algebra and $\mathbb{A}$ is an alter ego of $\boldsymbol{A}$. We say that $\mathbb{A}$ is of *finite type* in case it has only finitely many basic relations, partial operations, and operations. We also say that $\mathbb{A}$ has the *Interpolation condition* provided for every natural number n and every substructure $\mathbb{X}$ of $\mathbb{A}^n$, if φ is a structure preserving map from $\mathbb{X}$ into $\mathbb{A}$, then φ extends to an n-ary term function on $\boldsymbol{A}$. The second duality theorem arose in the work of Davey and Werner [9] while the duality compactness theorem is due to Willard (see [5,31]). Both theorems are fully formulated in [4], where proofs can also be found. We shall use the following combined formulation, which appears as Corollary 2.2.12 in [4].

The IC duality theorem. *Suppose* $\boldsymbol{A}$ *is a finite algebra. If* $\boldsymbol{A}$ *has an alter ego of finite type which satisfies the interpolation condition, then* $\boldsymbol{A}$ *is dualizable.*

So it remains to prove the following lemma.

Lemma 10. *Suppose G is an essential 4-transitive graph. Let n be a natural number, $\mathbb{X}$ be a substructure of $\mathbb{A}(G)^n$, and let φ be a structure preserving map from $\mathbb{X}$ into $\mathbb{A}(G)$. Then φ extends to a term function of $\boldsymbol{A}(G)$.*

That is, the interpolation condition holds for $\mathbb{A}(G)$. Now, invoking the IC duality theorem and Lemma 9, we obtain:

Corollary 11. *The graph algebra $\boldsymbol{A}(G)$ is dualizable for every finite graph G in which every connected component is either complete, bipartite complete, or a loose vertex.*

This establishes the implication (ii)⇒(i) of Theorem 1 which completes the proofs of Theorems 1 and 2.

In the proof below we will be showing that each structure-preserving map is the restriction of a term function. So let us briefly consider how term functions behave in graph algebras. Suppose t is a term. If v is a looped vertex, $t(v,\dots,v)=v$. Thus every looped vertex is in the range of t. If $t(v_0,\dots,v_{n-1})\neq 0$ (and t depends on all coordinates), then all the v_i's belong to the same connected component. Moreover, given t there is a coordinate j so that for all v_i we have $t(v_0,\dots,v_{n-1})$ is either v_j or 0. We will think of j as the dominant coordinate for t. We will establish the existence of a dominant coordinate for each structure preserving map.

Let us consider only graphs in which every connected component of the graph is either complete, bipartite complete, or a loose vertex. Baker et al. [2] provided standard forms for terms in such graph algebras. They showed that for these algebras, every term is equivalent to a term having one of the following forms:

(i) x (that is, a variable);

(ii) $x_0(x_0x_0)(x_1x_1)\dots(x_{n-1}x_{n-1})$ with the product associated to the left;

(iii) $x_0(y_0x_0)(y_1x_1)\dots(y_{n-1}x_{n-1})$ with the product associated to the left and $\{x_0,\dots,x_{n-1}\}$ disjoint from $\{y_0,\dots,y_{n-1}\}$.

In these standard forms the variables need not be distinct. For example, in (iii) above, while each x_i must be distinct from each y_j, it is permissible for some of the x_i's to be identical (this also applies to the y_j's). Note that if a term t is of type (ii), then t takes the value 0 off the union of the complete components, and such terms have only 0 and the union of the complete components in their ranges.

Keeping these standard forms in mind, as well as the earlier simple facts about graph algebra terms, will help the reader understand the steps we will take to establish the Interpolation Condition.

Proof of Lemma 10. We start with an easy observation.

Claim 1. *Every looped component is contained in the range of φ.*

Proof. If G has no looped component, we are done. Let L be a looped component and $v\in L$. Let α be the n-tuple so that $\alpha_i=v$ for all $i<n$. Since v is a constant of $\mathbb{A}(G)$, we have $\alpha\in\mathbb{X}$ and $\varphi(\alpha)=v$. □

We let $\mathrm{Rng}(\varphi)$ denote the range of φ. Since f and h are unary operations of $\mathbb{A}(G)$, our next claim below is clear.

Claim 2. *If D is a component such that $D \cap \mathrm{Rng}(\varphi)$ is nonempty, then $D \subseteq \mathrm{Rng}(\varphi)$.*

Let D be a component which has a nonempty intersection with $\mathrm{Rng}(\varphi)$. Let e be the designated vertex of D. Let $\gamma = \bigwedge_{\varphi(\beta)=e} \beta$, which certainly exists since $D \subseteq \mathrm{Rng}(\varphi)$. So $\gamma \in X$ since $\mathbb{X}$ is a substructure, and

$$\varphi(\gamma) = \varphi\left(\bigwedge_{\varphi(\beta)=e} \beta\right) = \bigwedge_{\varphi(\beta)=e} \varphi(\beta) = e.$$

We will call γ the *canonical n-tuple* of D. Observe that $\varphi(\alpha) = e$ if and only if $\gamma = \gamma \wedge \alpha$ if and only if $\gamma \leqslant \alpha$, for every $\alpha \in X$. So γ is the least element of the inverse image of a maximal element of $A(G)$ as a semilattice. We will call this the *basic property* of the canonical tuple γ.

Claim 3. *Each entry γ_i of the canonical n-tuple for D is either* 0 *or a vertex of D.*

Proof. Notice that $\varphi(k_D(\gamma)) = k_D(\varphi(\gamma)) = k_D(e) = e$, where e is the designated vertex of D. So by basic property of γ we have that $\gamma \leqslant k_D(\gamma)$. The claim follows. □

Let γ be the canonical tuple for D. We label the n coordinates as follows. We say coordinate i is *left* for D or *right* for D or *designated* for D if γ_i has that property. A coordinate i is *null* for D if and only if $\gamma_i = 0$. We will call i and j *opposite coordinates* if i is a left coordinate and j is a right coordinate or vice versa. We say a coordinate i is *dominant for D* provided $\varphi(\beta) = \beta_i$ for all $\beta \in X$ such that $\varphi(\beta) \in D$.

Claim 4. *If D is a component and $D \subseteq \mathrm{Rng}(\varphi)$, then there is a designated coordinate for D, and every designated coordinate is a dominant for D.*

Proof. Let γ be the canonical tuple for D and e be the designated vertex of D. By Claim 3 the entries of γ belong to the set $D \cup \{0\}$. Since $(D \cup \{0\}) - \{e\}$ is a unary relation of $\mathbb{A}(G)$ but $\varphi(\gamma) = e$, we deduce that some entry of γ must be e. Thus some coordinate is designated for D.

Now let i be any coordinate designated for D and suppose $\varphi(\beta) = v \in D$.

Suppose D is a looped component or a loose vertex. For $k=0$ or 1 we have $f^k(v)=e$. Let $f' = f^k$. Now f' respects components, and f' is one-to-one. But notice that

$$\varphi(f'(\beta)) = f'(\varphi(\beta)) = f'(v) = e.$$

From the basic property of γ we see that $\gamma \leqslant f'(\beta)$ and $e = \gamma_i \leqslant f'(\beta_i)$. Since e is maximal in the semilattice, $f'(\beta_i) = e$. Since f' is one-to-one, $\beta_i = v$. Consequently, $\varphi(\beta) = \beta_i$. This means that i is a coordinate dominant for D.

Suppose now that D is a loopless component with more than one vertex. Recall that $\varphi(\beta)=v$. If v is a left vertex, then $f^k(v)=e=p$ for $k=0$ or 1, and we take $f'=f^k$. If v is the right vertex, then $h(v)=e=p$ and we take $f'=h$. In either case v is the unique element such that $f'(v)=e$. The rest is similar to the preceding case. □

Claim 5. *If* $\mathrm{Rng}(\varphi)$ *contains some loopless component* P, *then either* φ *is the restriction to* X *of some projection or* P *has a right coordinate.*

Proof. We will suppose that P has no right coordinate and prove that φ must then be a restriction to X of some projection. Let γ be the canonical n-tuple for P. Now p is the designated vertex of P. Let $\beta=h(h(\gamma))$. Then

$$\varphi(\beta)=\varphi(h(h(\gamma)))=h(h(\varphi(\gamma)))=h(h(p))=p$$

and the entries of β come from $\{0,p\}$ since P has no right coordinates. So by the basic property of γ, we have that the entries of γ come from $\{0,p\}$.

Now let i be a designated coordinate for P, let $\alpha\in X$, and put $v=\varphi(\alpha)$. Notice that $\gamma\downarrow_p^v\alpha$ is defined (since the entries of γ come from $\{0,p\}$). Consequently,

$$\varphi(\gamma\downarrow_p^v\alpha)=\varphi(\gamma)\downarrow_p^v\varphi(\alpha)=p\downarrow_p^v v=p.$$

Since i is a dominant coordinate for P, we get $\gamma_i\downarrow_p^v\alpha_i=p$. But this means that $p\downarrow_p^v\alpha_i=p$. Therefore $\varphi(\alpha)=v=\alpha_i$. Because α was an arbitrary element of X, we conclude that φ is the restriction to X of the ith projection function. □

Claim 6. *There is a coordinate that is dominant for all components which are contained in* $\mathrm{Rng}(\varphi)$.

Proof. In view of the earlier claims, the only remaining case is that there is a loopless component $P\neq P_1$ contained in $\mathrm{Rng}(\varphi)$ and a looped component L. Any designated coordinate for L will be a dominant coordinate for L, and any designated coordinate for P will be dominant for P. Let γ be the canonical tuple for L, and let δ be the canonical tuple for P. We construe

$$\begin{pmatrix}\gamma\\ \delta\end{pmatrix}=\begin{pmatrix}\gamma_0, & \gamma_1, & \dots, & \gamma_{n-1}\\ \delta_0, & \delta_1, & \dots, & \delta_{n-1}\end{pmatrix}$$

as an n-tuple of ordered pairs (the ordered pairs being displayed vertically). Now

$$\varphi\begin{pmatrix}\gamma\\ \delta\end{pmatrix}=\begin{pmatrix}\varphi(\gamma)\\ \varphi(\delta)\end{pmatrix}=\begin{pmatrix}a\\ p\end{pmatrix}\notin R_{LP}.$$

Since φ preserves R_{LP}, then $\binom{\gamma}{\delta}\notin R_{LP}$. Thus $\binom{\gamma_i}{\delta_i}\notin R_{LP}$, for some $i<n$. Hence, $\gamma_i=a$ and $\delta_i=p$. Therefore, i is a designated coordinate for L as well as a designated coordinate for P. This means via Claim 4 that i is a coordinate dominant for all components. □

Claim 7. *Let L be a looped component and $\beta \in X$. Then $\varphi(\beta) \in L$ if and only if $\beta_i \in L$ for every coordinate i not null for L.*

Proof. Let γ be the canonical tuple for L. Since L is looped, $\varphi(\beta) \in L$ iff $a = a \cdot \varphi(\beta)$. Observe that

$$a \cdot \varphi(\beta) = \varphi(\gamma) \cdot \varphi(\beta) = \varphi(\gamma \cdot \beta).$$

Consequently, $\varphi(\beta) \in L$ iff $\varphi(\gamma \cdot \beta) = a$. Which, by the basic property of canonical tuples, is equivalent to $\gamma \leqslant \gamma \cdot \beta$, and this in turn is equivalent to $\gamma_i = \gamma_i \cdot \beta_i$ for each i which is nonnull for L. Hence, $\varphi(\beta) \in L$ iff $\beta_i \in L$ for every i not null for L. $\square$

Claim 8. *Let L be a looped component, $P \neq P_1$ be a loopless component with $P \subseteq \mathrm{Rng}(\varphi)$, and $\beta \in X$. Then $\varphi(\beta) \in L$ if and only if $\beta_i \in L$ for all i which are not null for P.*

Proof. Let γ be the canonical tuple for P. The remainder of the proof of this claim is like the proof of Claim 7 except that the operation $*$ is used in place of $\cdot$, the graph algebra multiplication, and the designated element p is used in place of the designated element a. $\square$

Claim 9. *Let $P \neq P_1$ be a loopless component with $P \subseteq \mathrm{Rng}(\varphi)$ and $\beta \in X$. Then $\varphi(\beta) \in P$ if and only if $\beta_k \in P$ for each k which is not null for P, and β_i and β_j belong to opposite parts of P if i and j are opposite coordinates.*

Proof. As usual, we let γ denote the canonical tuple for P. We claim that $\varphi(\beta)$ belongs to the right part of P iff for each nonnull k, we have β_k is in the part opposite to γ_k. Indeed, $\varphi(\beta)$ belongs to the right part of P iff $p = p \cdot \varphi(\beta)$. But $p \cdot \varphi(\beta) = \varphi(\gamma) \cdot \varphi(\beta) = \varphi(\gamma \cdot \beta)$. So $\varphi(\beta)$ belongs to the right part of P iff $\gamma \leqslant \gamma \cdot \beta$ iff for each nonnull k, we have $\gamma_k = \gamma_k \cdot \beta_k$ iff for each nonnull k, we have β_k is in the part opposite to γ_k.

On the other hand, $\varphi(\beta)$ belongs to the left part of P iff $p = p \cdot h(\varphi(\beta))$. But $p \cdot h(\varphi(\beta)) = \varphi(\gamma) \cdot \varphi(h(\beta)) = \varphi(\gamma \cdot h(\beta))$. This case now proceeds as before yielding the result that $\varphi(\beta)$ belongs to the left part of P iff for each nonnull k, we have β_k is in the same part as γ_k. $\square$

We can now describe a term function of $\boldsymbol{A}(G)$ which extends φ. There are four cases.

Case I: every component is disjoint from $\mathrm{Rng}(\varphi)$. In this case $\varphi(\beta) = 0$ for all $\beta \in X$, and there are no looped components. So we can take $x \cdot (x \cdot x)$ to be the desired term.

Case II: φ is the restriction of some projection function to X.

In this case, we can take our term to be x_j for some $j < n$.

Case III: φ is not the restriction to X of any projection function and $\mathrm{Rng}(\varphi)$ contains some component but no loopless component.

In this case, suppose the indices are arranged so that 0 is a dominant coordinate and $0,\ldots,m-1$ are the nonnull coordinates for the looped component. We can take our term to be $(\ldots((x_0(x_0x_0))(x_1x_1))\ldots)(x_{m-1}x_{m-1})$.

Case IV: φ is not the restriction to X of any projection function and $\mathrm{Rng}(\varphi)$ contains some loopless component.

In this case, we take our term to be $(\ldots(x(y_0z_0))(y_1z_1)\ldots)(y_{m-1}z_{m-1})$ where x occupies a dominant coordinate, the y's occupy the right coordinates, and the z's occupy the left coordinates with $z_0=x$; we allow variables to occur more than once in order to make the number of y's and the number z's come out the same.

That the term functions described above actually extend φ follows by Claim 1 (in Case I), Claims 6 and 7 (in Case III), and Claim 5, 6, 8 and 9 (in Case IV). □

We mentioned back on p. 167 that Baker et al. had produced standard forms for terms in graph algebras in which each connected component of the underlying graph is complete, bipartite complete, or a loose vertex. This particular fact is a corollary of the above proof. Each term function is certainly a structure preserving map. The standard form for the term is given in the four cases above.

Baker et al. used the standard form for terms as an aid to proving that each such algebra is finitely based. We used something slightly stronger than the standard form for terms to prove the dualizability of such algebras.

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